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Some Dynamical Properties for New System derived From Chen and T Systems

Hassan Kamil Jassim*, Ayed Elewis Hashoosh, Nabeel Jawad Hassan

Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq

Corresponding author e-mail: **hassan.kamil@yahoo.com**

ABSTRACT

In this paper we begin to study a nonlinear three dimensional continuous system with eight terms and three quadratic nonlinearities. The basic dynamical properties of the new system are analyzed by means of equilibrium point, stability and local bifurcations such as Pitchfork and Hopf bifurcations. Some of the basic dynamic behavior of the system is explored investigation in the Lyaponuv exponent, and we show that the new system is almost linear system.

Indexing terms/Keywords:

Chen system, T system, Hopf bifurcation, Stability, Lyaponuv exponent.

INTRODUCTION

The science of nonlinear dynamics has sparked many researchers to develop mathematical models that simulate vector fields of nonlinear chaotic physical systems. Nonlinear phenomena arise in all fields of engineering, physics, chemistry, biology, economics, and sociology [1,2,3].

Chen and Ueta [4] constructed a three-dimensional autonomous differential equation with only two quadratic terms, xy and xz. In fact, Chen system has been proved to be dual to the Lorenz system [7,10], many theoretical analysis and numerical simulation results in [4,12,13]. The Chen system [4,9] which takes the form:

$$\left.\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - xz + cy \\ \dot{z} &= xy - bz \end{aligned}\right\} \quad (1)$$

with parameters a, b and c are real parameters.

The T system has only two quadratic terms, xy and xz. The T system is related to the Lorenz, Chen, $L\ddot{u}$ systems [6] being a small generalization of the later one. The T system are:

$$\left.\begin{aligned}\dot{x}&=a(y-x)\\ \dot{y}&=(c-a)x-axz\\ \dot{z}&=xy-bz\end{aligned}\right\}\qquad(2)$$

with parameters a, b *and* c are real parameters.

Most researchers developed a new system depending on one chaotic system like Chen or T system. The proposed scheme in this paper based on merging two chaotic systems Chen chaotic system and T chaotic system. Therefore, will be added two systems in (1) and (2), thus a new system is shown in (3).

$$\left.\begin{aligned}\dot{x}&=2a(y-x)\\ \dot{y}&=2(c-a)x-(a+1)xz+cy\\ \dot{z}&=2xy-2bz\end{aligned}\right\}\qquad(3)$$

parameters a, b *and* c are real parameters and $a \neq 0, -1$.

TABILITY ANALYSIS

Proposition 1. If $\frac{b(3c-2a)}{a+1} > 0$, then the system (3) has three isolated equlibria

$O(0,0,0)$, $E_1(\sqrt{\frac{b(3c-2a)}{a+1}}, \sqrt{\frac{b(3c-2a)}{a+1}}, \frac{3c-2a}{a+1})$,

$E_2(-\sqrt{\frac{b(3c-2a)}{a+1}}, -\sqrt{\frac{b(3c-2a)}{a+1}}, \frac{3c-2a}{a+1})$

and for $\frac{b(3c-2a)}{a+1} \leq 0$ it has only one isolated equilibrium $O(0,0,0)$.

Proof: Solving the system

$$2a(y-x)=0 \Rightarrow x=y$$

$$2(c-a)x-(a+1)xz+cy=0 \Rightarrow x=0,\ z=\frac{2a-3c}{a+1}$$

$$2xy-2bz=0 \Rightarrow x=\pm\sqrt{bz}=\pm\sqrt{\frac{b(2a-3c}{a+1}}$$

which yields $x=0, y=0, z=0$ and for $\frac{b(2a-3c)}{a+1} > 0$ then

$x=y=\pm\sqrt{\frac{b(2a-3c)}{a+1}}$, $z=\frac{2a-3c}{a+1}$. Therefore, the system (3) has only one equilibrium $O(0,0,0)$ for $\frac{b(2a-3c)}{a+1} < 0$ but has three isolated equlibria: O, E_1, E_2 for all $\frac{b(2a-3c)}{a+1} > 0$.

Definition 1. [8] The constant critical point $\bar{x}$ of the nonlinear vector field $\dot{x} = f(x),\ \ x \in R^n$ is asymptotically stable if that all of the eigenvalues of Jacobian matrix $Df(\bar{x})$ have negative real parts.

Remark 1.[8] The constant critical point $\bar{x}$ is said to be unstable if at least one eigenvalues of $Df(\bar{x})$ has positive real part.

Lemma 1. For the system (3) the following statements are true:

(i) If $(a > c > 0, b > 0)$ *or* $(a > 0, b > 0, c < 0)$ then $O(0,0,0)$ is asymptotically stable.

(ii) If $(a < 0, c < 0)$ *or* $(a < 0, c > 0)$ *or* $(b < 0)$ *or* $(c > a > 0)$ then $O(0,0,0)$ is an unstable.

Proof: The Jacobian matrix of system (3) at the point $O(0,0,0)$ is:

$$J(O) = \begin{pmatrix} -2a & 2a & 0 \\ 2c-2a & c & 0 \\ 0 & 0 & -2b \end{pmatrix}$$

The characteristic polynomial of $J(O)$is :.

$$\lambda^3 + \lambda^2(2a+2b-c) + \lambda(4ab-6ac-2cb+4a^2) + 4ab(2a-3c) = 0.$$

Then, the eigenvalues of $J(O)$ are:

$$\lambda_1 = -2b,\ \lambda_2 = -a + \frac{1}{2}(c - \sqrt{c^2+20ac-12a^2}),\ \lambda_3 = -a + \frac{1}{2}(c + \sqrt{c^2+20ac-12a^2})$$

It is clear that if $b > 0,\ a > c > 0$ then $\lambda_1 < 0$ and $Re(\lambda_{2,3}) < 0$.

If $(a > 0, b > 0, c < 0)$ then $\lambda_1 < 0$ and $Re(\lambda_{2,3}) < 0$. Therefore, the point $O(0,0,0)$ is asymptotically stable.

If $b < 0$ then $\lambda_1 > 0$, if $c > a > 0$ then $\lambda_2 > 0$, and if $(a < 0, c < 0)$ *or* $(a < 0, c > 0)$ then $Re(\lambda_{2,3}) > 0$, consequently the point $O(0,0,0)$ is unstable.

Next, consider the stability of system (3) at

$$E_1\left(\sqrt{\frac{b(3c-2a)}{a+1}}, \sqrt{\frac{b(3c-2a)}{a+1}}, \frac{3c-2a}{a+1}\right), E_2\left(-\sqrt{\frac{b(3c-2a)}{a+1}}, -\sqrt{\frac{b(3c-2a)}{a+1}}, \frac{3c-2a}{a+1}\right)$$

For all $\frac{b(3c-2a)}{a+1} > 0$. Because the system is invariant under the transformation $(x,y,z) \to (-x,-y,z)$, one only needs to consider the stability of system (1) at $E_1\left(\sqrt{\frac{b(3c-2a)}{a+1}}, \sqrt{\frac{b(3c-2a)}{a+1}}, \frac{3c-2a}{a+1}\right)$.

Lemma 2. The equilibrium point $E_1\left(\sqrt{\frac{b(3c-2a)}{a+1}},\sqrt{\frac{b(3c-2a)}{a+1}},\frac{3c-2a}{a+1}\right)$ is asymptotically stable if and only if $2a+2b-c>0, bc>0, b(-4ac+2bc-c^2+4a^2)>0$.

Proof: Let

$$x = X + \sqrt{\frac{b(3c-2a)}{a+1}}$$

$$y = Y + \sqrt{\frac{b(3c-2a)}{a+1}}$$

$$z = Z + \frac{3c-2a}{a+1}$$

The system (3) becomes:

$$\left.\begin{aligned} \dot{X} &= 2a(Y-X) \\ \dot{Y} &= -cX-(a+1)XZ+cY-(a+1)\sqrt{\frac{b(3c-2a)}{a+1}}\,Z \\ \dot{Z} &= 2XY+2\sqrt{\frac{b(3c-2a)}{a+1}}\,(X+Y)-2bZ-\frac{2b(3c-2a)}{a+1} \end{aligned}\right\} \quad (4)$$

Hence, one has to consider the stability of system (4) at $O(0,0,0)$.

The Jacobian matrix of system (4) at the point $O(0,0,0)$ is:

$$J = \begin{pmatrix} -2a & 2a & 0 \\ -c & c & -(a+1)\sqrt{\frac{b(3c-2a)}{a+1}} \\ 2\sqrt{\frac{b(3c-2a)}{a+1}} & 2\sqrt{\frac{b(3c-2a)}{a+1}} & -2b \end{pmatrix}$$

The characteristic polynomial of J is :

$$\lambda^3+(2a+2b-c)\lambda^2+4bc\lambda+8ab(3c-2a)=0 \quad (5)$$

Then, from Routh-Hurwits conditions, this equation has all roots with negative real parts if and only if $A>0, C>0$ and $AB-C>0$ where

$A=2a+2b-c, B=4bc, C=8ab(3c-2a)$, that is:

$$\left.\begin{aligned} &2a+2b-c>0 \\ &ab(3c-2a)>0 \\ &b(4a^2-c^2+2bc-4ac)>0 \end{aligned}\right\} \quad (6)$$

PITCHFORK AND HOPF BIFURCATIONS

Proposition 2. If $a=\frac{3c}{2}$ the equilibrium point $O(0,0,0)$ of the system (3) undergoes a pitchfork bifurcation.

Proof: If $a\geq\frac{3c}{2}$ the equilibrium point $O(0,0,0)$ is asymptotically stable and if $a<\frac{3c}{2}$ equilibrium point $O(0,0,0)$ unstable and two equlibria birth

$E_1(\sqrt{\frac{b(3c-2a)}{a+1}},\sqrt{\frac{b(3c-2a)}{a+1}},\frac{3c-2a}{a+1}),E_2(-\sqrt{\frac{b(3c-2a)}{a+1}},-\sqrt{\frac{b(3c-2a)}{a+1}},\frac{3c-2a}{a+1})$ are stable. Therefore, the system (3) has pitchfork bifurcation at the $a=\frac{3c}{2}$, where $O(0,0,0)$ is bifurcation point .

Remark 2. The equilibrium point $O(0,0,0)$ has no conjugate pair of pure imaginary eigenvalues, thus Hopf bifurcation does not occur at this point.

Remark 3. The characteristic equation associated with the linearization of system (4) at the equilibrium:

$$E_1(\sqrt{\frac{b(3c-2a)}{a+1}},\sqrt{\frac{b(3c-2a)}{a+1}},\frac{3c-2a}{a+1}),E_2(-\sqrt{\frac{b(3c-2a)}{a+1}},-\sqrt{\frac{b(3c-2a)}{a+1}},\frac{3c-2a}{a+1})$$

is given by:

$$\lambda^3+(2a+2b-c)\lambda^2+4bc\lambda+8ab(3c-2a)=0 \tag{5}$$

It is easy to see that this equation has no zero root since $ab(3c-2a)\neq 0$, thus we only consider the Hopf bifurcation .

Let b be the Hopf bifurcation parameter. Suppose that $b=b_0=\frac{4a^2+4ac+c^2}{4c}$. Then the equilibrium $E_1(E_2)$ has the following eigenvalues:

$$\lambda_1=-\frac{4a^2+6ac}{c},\ \lambda_{2,3}=\mp i\sqrt{8a^2+8ac+2c^2}\ .$$

Theorem 1. If $b=b_0=\frac{4a^2+4ac+c^2}{4c}$, then the system (4) undergoes a Hopf bifurcation at the equilibrium point $E_1(\sqrt{\frac{b(3c-2a)}{a+1}},\sqrt{\frac{b(3c-2a)}{a+1}},\frac{3c-2a}{a+1})$.

Proof: If $b=b_0=\frac{4a^2+4ac+c^2}{4c}$, then the equation (5) becomes

$(\lambda+\frac{4a^2+6ac}{c})(\lambda^2+\sqrt{8a^2+8ac+2c^2})=0$. Therefore, characteristic equation has a negative real root $\lambda_1 = -\dfrac{4a^2+6ac}{c}$ and apair of purely imaginary roots

$\lambda_{2,3} = \mp i\sqrt{8a^2+8ac+2c^2}$.

Differentiating both sides of equation (5) with respect to b, we obtain

$$3\lambda^2\frac{d\lambda}{db} + 4a\lambda\frac{d\lambda}{db} + 4b\lambda\frac{d\lambda}{db} + 2\lambda^2 - 2c\lambda\frac{d\lambda}{db} + 4bc\frac{d\lambda}{da} + 4c\lambda + 24ac - 16a^2 = 0$$

$$\frac{d\lambda}{db} = -\frac{2\lambda^2 + 4c\lambda - 16a^2 + 24ac}{3\lambda^2 + (4a+4b-2c)\lambda + 4bc}$$

by setting $b = b_0$ and $\lambda = i\sqrt{4bc}$

$$\lambda_b'(b_0) = \frac{2b_0c + 4a^2 - 6ac - ic\sqrt{bc}}{2b_0c - i(2a+2b-c)\sqrt{bc}}$$

$$Re[\lambda_b'(b_0)] = \frac{6b_0c + 8a^2 - 10ac - c^2}{4b_0c + (2a+2b-c)^2} \neq 0.$$

Therefore, $\lambda_b'(b_0) \neq 0$. According to Hopf bifurcation theorem in [11], the system (4) has display a Hopf bifurcation at $(0,0,0)$, so the system (3) display a Hopf bifurcation at the point

$$E_1\left(\sqrt{\frac{b(3c-2a)}{a+1}}, \sqrt{\frac{b(3c-2a)}{a+1}}, \frac{3c-2a}{a+1}\right).$$

Definition 2. [1] Let V be a subset of R^3 that contains the origin in its interior. Assume that F, G and H are real valued functions on V that vanish at the origin and whose partial derivatives are continuous and also vanish at the origin. Then the system

$$\begin{aligned}\dot{x} &= a_1x + b_1y + c_1z + F(x,y,z)\\ \dot{y} &= a_2x + b_2y + c_2z + G(x,y,z)\\ \dot{z} &= a_3x + b_3y + c_3z + H(x,y,z)\end{aligned}$$

is almost linear system.

Proposition 3. the system (3) is almost linear system at the point $O(0,0,0)$.

Proof: we write the system (3) as follows:

$$\begin{aligned}\dot{x} &= 2ay - 2ax + F(x,y,z)\\ \dot{y} &= (2c-2a)x + cy + G(x,y,z)\\ \dot{z} &= -2b\,z + H(x,y,z)\end{aligned}$$

where $F(x,y,z)=0, G(x,y,z)=-(a+1)xz$ and $H(x,y,z)=2xy$. Then $F(0,0,0)=G(0,0,0)=H(0,0,0)=0$, and all first partials of F, G and H continuous and vanish at the point $O(0,0,0)$. Therefore the system (3) is almost linear system at the point $O(0,0,0)$.

Remark 4. The statement $\frac{b(3c-2a)}{a+1}>0$ it means that:

(i) $(a>-1, b>0, c>\frac{2a}{3})$

(ii) $(a<-1, b>0, c<\frac{2a}{3})$

(iii) $(a<-1, b<0, c>\frac{2a}{3})$

(iv) $(a>-1, b<0, c<\frac{2a}{3})$

LYAPONUV EXPONENT

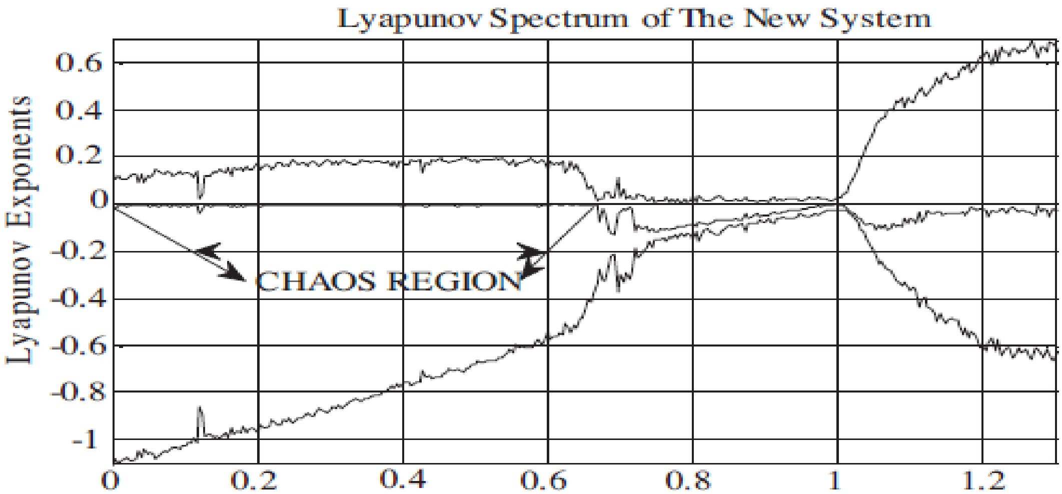
To compute the maximal Lyaponuv exponent of a system of ordinary differential equations we must integrate both the original system and its linearization $\dot{v}=A(t)v$. Essentially any initial vector v_0 can be used because almost all vectors will have some component along the direction of the maximal Lyaponuv direction. We can not compute the limit in maximal Lyaponuv exponent $\gamma(x,v)=\lim_{t\to\infty}\sup\frac{1}{t}|\Phi(t,x)v|$ but instead simply integrate for some long time T and estimate: $\gamma_{\max}(T)=\frac{1}{T}Ln\frac{|v(T)|}{|v_0|}$ such that $v_0\neq 0$.This quantity will rapidly converge to the maximal exponent; to estimate the error in the computation, it is useful to plot $\gamma_{\max}$ as a function of T [5].

Next, we calculate maximal Lyaponuv exponent for system (3) at the critical points by using Matlab program.

1. The maximal Lyaponuv exponent for system (3) at the critical point $O(0,0,0)$with parameters $(a>c>0, b>0)$ and $(a>0, b>0, c<0)$ is negative number.
2. The maximal Lyaponuv exponent for system (3) at the critical point $O(0,0,0)$with parameters$(a<0, b<0, c<0)$ and $(a>0, b>0, c>0)$ and $(a>0, b<0, c<0)$ and $(a>0, b<0, c>0)$ is positive number.
3. The maximal Lyaponuv exponent for system (3) at the critical point $E_1(\sqrt{\frac{b(3c-2a)}{a+1}}, \sqrt{\frac{b(3c-2a)}{a+1}}, \frac{3c-2a}{a+1})$ with parameters $(a>-1, b>0, c>\frac{2a}{3})$ is negative number.
4. The maximal Lyaponuv exponent for system (3) at the critical point

$E_1(\sqrt{\frac{b(3c-2a)}{a+1}},\sqrt{\frac{b(3c-2a)}{a+1}},\frac{3c-2a}{a+1})$ with parameters $(a<-1,b>0,c<\frac{2a}{3})$ and $(a<-1,b<0,c>\frac{2a}{3})$ and $(a>-1,b<0,c<\frac{2a}{3})$ is positive number.

Figure (1) shows the Lyaponuv spectrum of the new system (3) for a varying parameter a, and constant parameter $b=0.5$ and $c=0.3$. As can be seen from the Lyaponuv exponents spectrum, when a is in the range (0, 0.665), the new system is chaotic with a positive Lyaponuv exponent.



Finger (1) Lyapunov spectrum of the new chaotic system

for varying parameter a, and constant parameters $b=0.5$ and $c=0.3$.

CONCLUSIONS

In this paper we further investigated a nonlinear differential system with three equilibrium points, origin $O(0,0,0)$ and another two E_1,E_2. In the origin, the system (3) displays a pitchfork bifurcation and in the other two equilibrium points a Hopf bifurcation. The system (3) is almost linear system at the point $O(0,0,0)$. The system (3) has a positive Lyaponuv Exponent in a special cases. Surely, there is still a lot of work, and this paper is a step in analyzing this system.

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